DETERMINING THE FRACTAL DIMENSION OF A TIME SERIES WITH A NEURAL NET

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1. INTRODUCTION

There are several methods for estimating the fractal dimension of a time series of data such as the box counting method and the correlation method [DeGrauwe, Dewachter and Embrechts, 1993] [Peitgen and Saupe, 1988]. The application of these methods are often demanding in computing time and require expert interaction for interpreting the calculated fractal dimension. Artificial neural nets (ANN) offer a fast and elegant way to estimate the fractal dimension of a time series. A backpropagation net was trained to find the fractal dimension of a time series with encouraging results. Training patterns of time series with known fractal dimension were generated with the fractal interpolation method described by Barnsley [Barnsley, 1988]. Two artificial neural nets were trained with the backpropagation algorithm on the amplitude spectra of fractal signals. One ANN used local normalization, the other ANN used global normalization of the spectral components. The trained neural nets were then tested on entirely different fractal signals generated with the Weierstrass-Mandelbrot function [Mandelbrot, 1983]. Both ANN's could estimate the fractal dimension for other fractal time series generated with the fractal interpolation technique, but only the ANN with global normalization could correctly identify the fractal dimension of the Weierstrass-Mandelbrot based time series (within 10 percent error).

Section 2 briefly reviews fractal interpolation. Section 3 summarizes how amplitude spectra can be generated from fractal interpolation graphs and explains the difference between local and global normalization of power spectra. Section 4 introduces the Mandelbrot-Weierstrass function, and provides details of the neural net architecture. The failure of the first ANN to correctly identify Weierstrass-Mandelbrot functions led to a deeper insight into the relationship between fractal time series and their amplitude spectra, and is discussed in Section 5.

2. FRACTAL INTERPOLATION

Barnsley [Barnsley, 1988] introduced the *fractal interpolation* method which applies *iterated function systems* (*IFS*) to produce a fractal (*graph*) with a known *fractal dimension* through N + 1 points. Fractal interpolating generates a graph which is the attractor of an iterated function system of N *contractive affine transformations (or shear transformations).* The N shear transformations are of the type:

$$\begin{pmatrix} X^{NEW} \\ Y^{NEW} \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} X^{OLD} \\ Y^{OLD} \end{pmatrix} \begin{pmatrix} e_n \\ f_n \end{pmatrix}$$

where $1 < n < N$. Starting from a randomly chosen initial value for X and Y, a transformation (of the N possible transformations) is selected at random and applied to X and Y leading to new values for X and Y. A new transformation is randomly picked, and the recursive process is repeated many times. The successive values for X and Y are plotted (e.g. Fig. 1c) and form the graph of the fractal interpolation curve. Figure 1 illustrates this process, where 10,000 points of the fractal interpolation graph were generated through (N=6) original points with respective dimensions of 1.1 (Fig. 1a), 1.5 (Fig. 1b), and 1.8 (Fig. 1c). Note that the original points lie on the fractal interpolation graphs, and that Figs. 1a-c provide different graphs through the same original points. The coefficients a_n, c_n, d_n, e_n, f_n are generated from the x and y coordinates of N + 1 equally spaced data points, (x_0, y_0), (x_1, y_x), . . . , (x_N, y_N), by applying the following equations:

$$a_n = \frac{(x_n - x_{n-1})}{(x_N - x_o)} \qquad e_n = \frac{(x_N\, x_{n-1} - x_o x_n)}{(x_N - x_o)}$$

$$c_n = \frac{(y_n - y_{n-1})}{(x_N - x_o)} - d_n \frac{(y_N - y_o)}{(x_N - x_o)} \qquad f_n = \frac{(x_N\, y_{n-1} - x_o x_n)}{(x_N - x_o)} - d_n \frac{(x_N\, y_o - x_o y_N)}{(x_N - x_o)}$$

The d_n parameters are called the *scaling factors* of the shear transformations and are free parameters which are related to the fractal dimension, D, of the interpolating graph according to

$$D = 1 + \frac{\log\left(\sum_{n=1}^{N} |d|_n\right)}{\log(N)}$$

if the following restrictions hold:

a. the N + 1 data points are equally spaced,
b. $\sum_{n=1}^{N} |d_n| > 1$
c. $|d_n| < 1$ for $1 < n < N$.

These conditions still leave some freedom for choosing the values of the scaling factors for a given fractal dimension. In our evaluations the scaling factors, d, were initially chosen as either +0.5 or -0.5 (where the sign was selected randomly), and then adjusted by multiplying all the d factors by the same factor to obtain a desired fractal dimension.

Eight hundred (800) IFS graphs, each containing 10,000 points were prepared with this technique as follows:

a. Choose 6 (X,Y) points which will form the basic points through which the fractal graphs will interpolated. The X coordinates were fixed for all the 800 graphs to X=1,2,3,4 and 5. The Y coordinates were generated as a random number between 1 and 2. Note that for six basic points the N used in the previous formulas equals 5.
b. The parameters a_n, c_n, e_n, f_n (for $1 < n < 5$) are calculated based on the equations for the shear transformation parameters.
c. The scaling parameters are initially set to +0.5 or -0.5, and the fractal dimension, D, is randomly chosen between 1 and 2. The scaling parameters are then adjusted by multiplying them with a factor corresponding to the chosen fractal dimension.
d. The IFS system is applied to an initial point (X, Y), and 10,000 points are calculated by repeatedly applying one of the five shear transformation to the new points.

3. PREPROCESSING THE TRAINING PATTERNS

In order to represent the time series data in a compact way to the neural network, power spectra were calculated for each fractal interpolation graph. Two different ways of presenting the power spectral information were attempted:

(a) Bin averaged amplitude spectra with the largest coefficient of each bin scaled to unity, and the other coefficients scaled appropriately (i.e. *local normalization*).
(b) Selected spectral components of the amplitude spectrum (components 10 through 49 of a FFT based on 1,024 points) with all the spectra scaled by the same scaling factor in order to have appropriate values for the ANN (i.e. *global normalization*).

Because the data in the fractal interpolation graphs are not ordered or equally spaced, time series were generated by interpolating 2,048 and 1,024 equally spaced data points for cases (a) and (b) respectively. The amplitude spectra of the time series for case (a) were calculated based on the fast Fourier transform and bin averaging the spectral components over 18 bins (with an equal spacing on the log frequency scale). The bin averaged spectra corresponding to Figs. 1a-c are shown in Figs. 1d-f on a double logarithmic scale. Note that the spectral components are actually the bin averaged amplitudes of the power spectrum (and not bin averaged squares of the amplitudes). We referred therefore to these plots as *amplitude spectra* (as opposed to power spectra). Note from the least squares line fits in Figs. 1d-f that the amplitude spectra generally have a linear downward trend on a double logarithmic scale. The relationship between the exponent of the amplitude spectrum, b, and the fractal dimension, D, is often cited in the literature [Mareschal, 1989] as

$$D \approx -(5/2 - b)$$

This relationship has been proven for the Weierstrass-Mandelbrot function, but it has not been demonstrated that this conjecture can be extended to estimate the fractal dimension of any fractal signal. The fractal dimension estimated from this expression is 1.3, 1.51 and 1.82 for the curves of Fig. 1d, 1e and 1f respectively. Experimentally, we find that the relationship between the exponent of the amplitude spectrum and the fractal dimension breaks down for fractal interpolation graphs when the fractal dimension approaches unity, but seems to hold for higher fractal dimensions. The neural net will therefore have to do more than learning to estimate the slope of an amplitude spectrum on a double logarithmic scale. The first sixteen (out of eighteen possible) amplitude spectral components were retained for training the neural net. The preparation of the training patterns for the neural net can be broken down in the following steps:

e. Order the 10,000 data generated by the fractal interpolation IFS system.
f. Interpolate (2,048 for case a and 1,024 for case b) equally spaced data and calculate amplitude spectra.
g. Bin average the spectra and normalize locally for the first ANN, select components 10-49 and do a global normalization after all the spectra have been generated for the second ANN.
f. The fractal dimension for a time series is a number between 1 and 2, so subtracting unity from the fractal dimension provides a proper scaling for the neural net.

The time series and the amplitude spectral components (10 through 49) are shown for the Mandelbrot-Weierstrass function in Figs. 2a-b, a fractal interpolation graphs and the corresponding amplitudes of the amplitude spectra can be found in Figs. 2c-d.

4. PREPROCESSING TEST PATTERNS AND TRAINING THE ANNs

The testing patterns were based on an entirely different type of fractal with well known fractal dimension, the *Mandelbrot-Weierstrass function*[Mandelbrot, 1983]. This function is described by

$$W(x) = \sum_{i=-\infty}^{\infty} \lambda^{(D-2)i}\left(1-\cos\left(\lambda^{i}x\right)\right)$$

where $\lambda > 1$ and $1 < D < 2$. This function is numerically verified to have a fractal dimension D, and has the scaling property

$$W(kx) = k^{2-D}\, W(x) \qquad \text{for all } x$$

In these tests we used $\lambda = 5$ and varied i ranging from -20 to 20.

Two different neural nets were trained and tested. The first neural net is a 16x9x7x1 net (2 hidden layers), and was trained with 800 locally normalized bin averaged amplitude spectra to an error of 4.7 percent on the fractal dimension (Fig. 3a). The net was first tested on 800 fractal interpolation curves, yielding an average error of 9.5 percent on the fractal dimension (Fig. 3b). The second set of test patterns was generated with the Mandelbrot-Weierstrass function. The ANN results show a regularly curved shape on a scatter plat with little variance in the results, but a large systematical error can be noted (Fig. 3c).

The second neural net is a 40x5x2x1 backpropagation net and was trained with 500 globally normalized amplitude spectral data (corresponding to frequencies 10-49 of the FFT). The training and testing results on fractal interpolation time series were qualitatively similar to the first neural net, but the results of the fractal dimension based on the Weierstrass-Mandelbrot function are now acceptable (less than 10 percent error, Fig. 3d).

5. INTERPRETATION OF THE RESULTS

It is clear from the results of the ANNs that a neural net can be trained to learn the fractal dimension of a time series. We did not succeed in training neural nets to errors that are smaller than 5 percent in the fractal dimension of the test set. With diligence and further experimentation of neural net structures and training strategies, this error can be further reduced.

The interesting observation here is that even when a net can be trained to recognize the fractal dimension of a time series, the generalization to very different (but also fractal) time series depends on the way the spectral data were preprocessed.

With local normalization the neural net learns how the fractal dimension is related to the slope, b, of the amplitude spectrum. The theoretical relationship between D and b has only been proven to hold for the Mandelbrot-Weierstrass function [Mareschal, 1989]. Further experimentation revealed that actual amplitude spectra based on a finite number of frequencies do not follow the theoretical expression when D comes closer to unity. This observation is a confirmation of similar results in previous other publications [Fox, 1989]. Different types of fractals will furthermore yield different relationships between the slope of the amplitude spectrum and the fractal dimension. This explains why a neural net that has learned to correlate the slope of one type of fractal with the fractal dimension, cannot use that relationship to estimate the dimension of a different type of fractal. The regularity of the test results of the Weierstrass-Mandelbrot function is consistent with this observation and the systematic error observed in Fig. 2c can thus be explained.

Local normalization poses a second difficulty for fractal time series. Because the X-axis (often representing time), and the Y-axis of a time series generally do not have the same physical meaning, a time series is actually not self-similar but self-*affine*. This property makes the interpretation of the fractal dimension more difficult. A scaling of the Y data will have an effect on the fractal dimension. Local normalization in the Fourier domain implies that the Y-values of different fractal time series are scaled differently, which will effect the perceived fractal dimensions.

It is in view of this second comment that we considered global normalization in the second ANN. The neural net can now correctly identify the fractal dimension of a time series generated with the Mandelbrot-Weierstrass function. Note that the amplitude spectra of the fractal interpolation time series and the Weierstrass-Mandelbrot time series are very different (Figs. 3b and 3d). The amplitude spectra of the Weierstrass-Mandelbrot function have a generally linear scaling on a double logarithmic plot, but have several peaks in them. One of these peaks can be observed in the frequency range of Fig. 3b. The amplitude spectra of the fractal interpolation curves are more irregular. It is obvious here that the neural net did more than just learning the slope of the amplitude spectra.

6. CONCLUSIONS

This paper shows that neural nets can be trained for recognizing the fractal dimension of a time series. In order for this method to be general it is imperative that the amplitude spectra are globally normalized. The results are encouraging and well trained neural nets might ultimately lead to extremely fast and accurate estimates of the fractal dimension of a time series.

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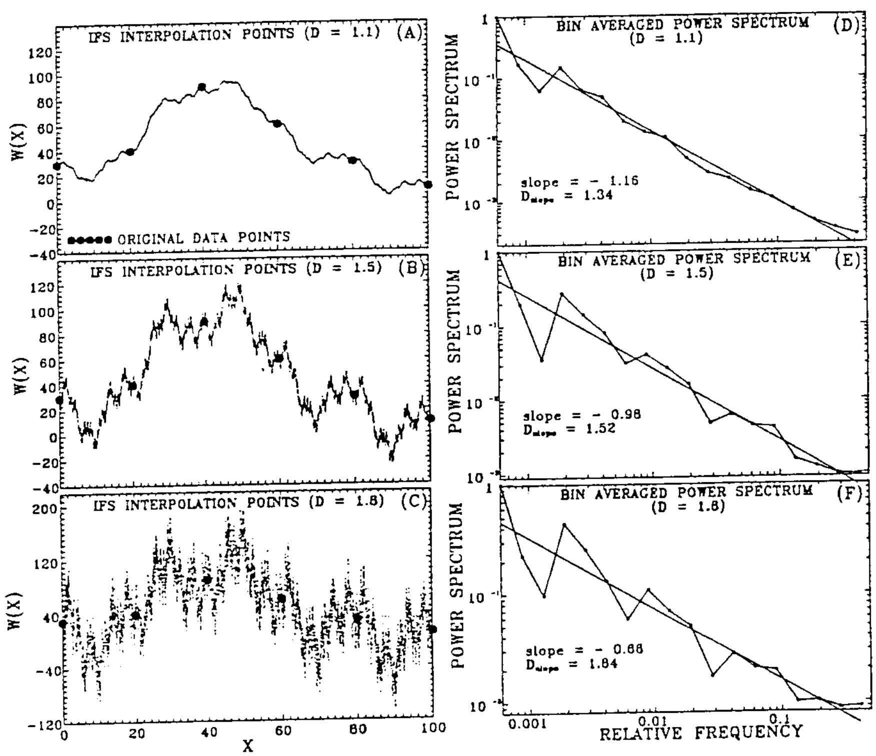

Figure 1 Examples of graphs generated with Barnsley's fractal interpolation method and the corresponding bin averaged amplitude spectra.

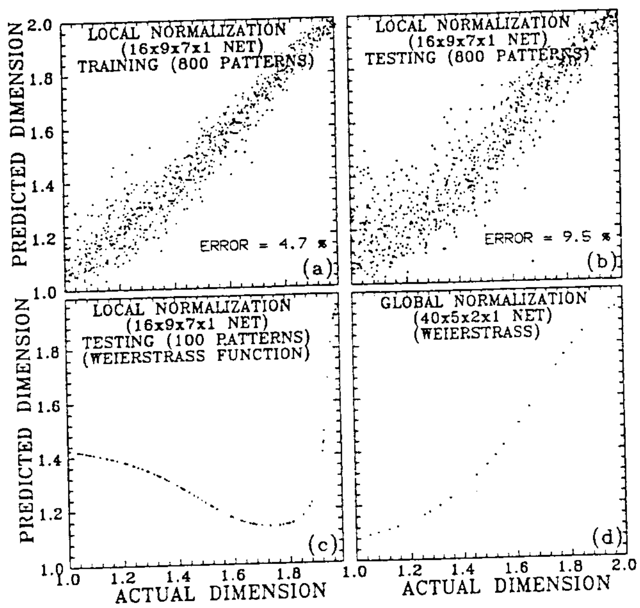

Figure 2 Scatterplot for training and testing results of the ANN with global and local normalization of the amplitude spectra.

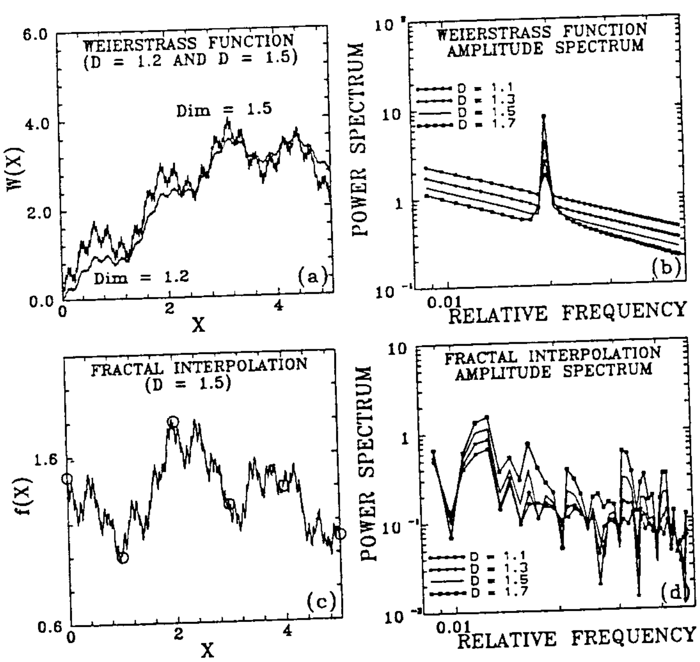

Figure 3 Examples of the Weierstrass-Mandelbrot function and fractal interpolation graphs and the corresponding amplitude spectral components (10th through 49th frequency).